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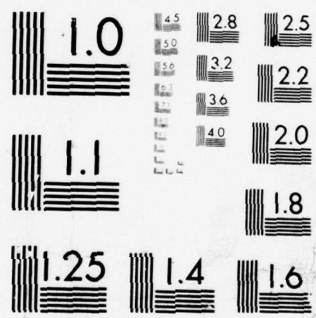

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ON THE RELATIONSHIP OF SOME RESULTS OF GELFAND-DIKII AND P. MOERBEKE, AND A NATURAL TRACE FUNCTIONAL FOR FORMAL ASYMPTOTIC PSEUDO-DIFFERENTIAL OPERATORS

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ABSTRACT 10 M./Adler

This paper developed out of an attempt to understand results of Gelfand-Dikii [1] and P. Moerbeke (unpublished version of [2]) in a unified way. We provide a natural abstract setting for understanding the symplectic structure involved in both results. The setting is an orbit in the dual algebra of a group. We also discuss a natural trace functional for formal asymptotic pseudo-differential operators. In addition we discuss so-called Lenard recursion relations inherent in these structures.

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SIGNIFICANCE AND EXPLANATION

It is one thing to write down the equations governing the behavior of a physical system in time; it is quite another thing to solve them. Historically, an important step forward in explaining the behavior of mechanical systems was made when it was realized that, for some systems, certain quantities like energy and angular momentum were independent of time, i.e. invariant. This is a tremendous help in solving the equations both analytically and numerically, since the invariants are equivalent to first integrals of the equations of motion, and constrain the class of functions within which we need look for solutions.

It turns out that in many problems of physical interest involving partial as well as ordinary differential equations, it is possible to find a denumerable number of quantities that are invariant as time evolves. This paper gives a unified method for finding invariants for a class of equations that includes crystal lattices (the Toda equations), stochastic processes (Kac-Moerbeke), and water waves (Boussinesq and Korteweg-deVries). In addition, the mechanical structure of these systems, as discovered by respectively Gelfand-Dikii, and P. Moerbeke are given geometrical significance.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ON THE RELATIONSHIP OF SOME RESULTS OF GELFAND-DIKII AND P. MOERBEKE, AND A NATURAL TRACE FUNCTIONAL FOR FORMAL ASYMPTOTIC PSEUDO-DIFFERENTIAL OPERATORS

M. Adler

1. Introduction

This paper developed out of an attempt to understand the results of Gelfand-Dikii [1] and P. Moerbeke (unpublished version of [2]), in a unified way. The above authors respectively discovered generalizations of the Korteweg-deVries equation and Toda system equations, which are completely integrable Hamilton systems whose equations of motion are expressible in terms of the Lax-isospectral equation.

The crucial observation in both cases is that the relevant symplectic structure in these problems is the orbit symplectic structure of Kostant-Kirillov [12,18]. We understand that for the Toda system, this fact was known to Kostant*. We observe that this symplectic structure plays a role in the n-dimensional Euler spinning top problem of V. Arnold [11], along with an isospectral equation, as was discovered by L. Dikii [15]. In addition, quotient symplectic structures through Hamiltonian group actions [12], of which the Kostant-Kirillov construction is but an example, have been seen to play a role in the Moser-Calogero systems, see [13,14].

In the case of the generalized Toda systems, the relevant group G is the group of lower triangular matrices with nonzero diagonal elements. We identify it's dual algebra $\mathcal{L}^*$ with the upper triangular matrices through the trace form, and then the orbit Hamiltonian phase space θ_A of interest is of the form

$$\theta_A = \{[U^{-1}AU]_+ | U \in G\}, \quad A \in \mathcal{L}^* ,$$

where $[B]_+$ denotes the matrix formed from B by setting its lower triangular entries equal to zero. We shall wish to place certain conditions on A, see Section 2.

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*Personal communication from J. Moser.

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For the case of the generalized Korteweg-deVries equation, the relevant group G is the formal pseudo-differential symbols of negative type, translated by the identity element 1, whose dual $\mathcal{L}^*$ we may identify, through a trace form to be specified, with the differential symbols of nonnegative type, which are identified with differential operators. Then the Hamiltonian orbit space of interest is of the same form as θ_A above, where the operation $[\ ]_+$ shall denote the natural projection of a pseudo-differential symbol onto the nonnegative symbols. For the case of the Korteweg-deVries equation, $A = -D^2 + q$, an orbit θ_A is, roughly speaking, specified by the condition $\int q dx = \text{constant}$. We also point out that although the formula "in coordinates" for the correct symplectic structure of the generalized Korteweg-deVries equations appears in [1], it comes about through strictly computational procedures, and it is our purpose to place the formula in its proper geometric setting. This also necessitates redoing parts of [1] in a less ad hoc procedure than was employed in [1], so as to give complete and consistent proofs, and moreover simplify proofs in general that appear there.

We amplify the previous paragraph. The algebra of pseudo-differential operators has a subalgebra consisting of those symbols which have an asymptotic description at ∞. One may think of taking this description as a set of normal corrdinates for the subalgebra, since one identifies symbols modulo their behavior on compact sets. This suggests that one should abstract the algebraic content of the above situation, and just work in a formal setting. Namely we work with formal Laurant series in a variable ξ over some formal differential ring, with the multiplication rule inherited from pseudo-differential operator theory. For motivation, one thinks of the formal Laurant series as being in a real variable ξ embedded in a complex neighborhood of ∞, or more precisely as a series germ. We then define the trace functional (which is of course commutative) as the equivalence class of the coefficient of the ξ^{-1} term, where we identify two elements precisely if their difference is a total derivative of a ring element. The trace functional makes possible the identification of a cotangent bundle, and from then on, the Korteweg-deVries case proceeds in perfect analogy with the Toda

system case, a point which we wish to stress. The method also works automatically for the self-adjoint case, which merely involves a slightly different coordinization of the formal Laurant series then the previous case, or for the case where the relevant ring is composed of square matrices (see [19]) whose entries are themselves differential ring elements. In the latter case, we must take as the trace functional, the equivalence class of the matrix trace of the coefficient of the ξ^{-1} term.

We point out that if the semi-lower triangular matrices have a place in the Fourier theory of say Z, or Z_n, the integers and the integers modulo n, respectively. There place is analogous to the role of the formal asymptotic pseudo-differential operators of negative type. We discuss this briefly at the end of Section 2.

We conclude in Section 4 with a generalized set of so-called Lenard relations, which provides, at least in theory, an alternate method of computing all the relevant quantities in the theory, and in addition, in the case of self-adjoint operators, the relations contain formal spectral information concerning the operators studied. These relations are essentially a consequence of the law of exponents for operators. There are a number of conjectures concerning these relations. We note that Bill Symes has discussed recursion relations in [8], and the author in [5].

I wish to thank Bill Symes for many stimulating discussions. I am also indebted to J. Moser who provided the stimulus and encouragement for this line of work, along with the clarification of some points. In addition, my thanks extend to Joel Robbin and C. Conley, for helpful suggestions, and at whose seminar at the University of Wisconsin, Madison, these results were first presented. We also mention that Bill Symes has a generalization of the Gelfand-Dikii theorem in the setting of this paper.

II. The Generalized Toda Systems

As motivation, and of interest in its own right, we first discuss the Toda system and its generalizations, which display the relevant structures in both examples.

To this end we introduce the group G of lower triangular matrices with strictly nonzero diagonal entries. Its Lie algebra $\mathcal{L}$ is just the lower triangular matrices, and we identify $\mathcal{L}^*$, the dual of $\mathcal{L}$, with the upper triangular matrices (see [18]) via the trace, $\langle \cdot \rangle$, functional and associated pairing $\langle \cdot,\cdot \rangle$,

(2.1) $$\langle A \rangle = \mathrm{tr}[A_{ij}] = \Sigma A_{ii}, \quad \langle E,F \rangle = \langle EF \rangle .$$

Since $g \in G$ acts on $\mathcal{L}$ via conjugation, it naturally acts on $\ell^* \in \mathcal{L}^*$ through duality, and with our given representation of $\mathcal{L}^*$ its action is described by

(2.2) $$g : \ell^* \to [g^{-1}\ell^* g]_+ ,$$

where $[\ ]_+$ denotes the projection operation of setting all terms below the diagonal equal to zero. We denote an orbit of this action by $\theta = \theta_{\ell^*} = \{[g^{-1}\ell^* g]_+ | g \in G\}$. From (2.2), the tangent space of θ at ℓ^* is described by

(2.3) $$T\theta_{\ell^*}(\ell^*) = \{[\ell,\ell^*]_+ | \ell \in \mathcal{L}\} ,$$

and the natural symplectic 2-form ω, of Kostant-Kirillov [12], associated with the orbit space θ is defined via

(2.4) $$\omega([\ell^*,\ell_1]_+,[\ell^*,\ell_2]_+)(\ell_*) = \langle \ell^*,[\ell_1,\ell_2] \rangle = \langle [\ell^*,\ell_1]_+,\ell_2 \rangle .$$

If we let $A = [A_{ij}]$ be the running variable on $\mathcal{L}^*$, and $H = H(A) = H([A_{ij}])$ be a function on $\mathcal{L}^*$, we may uniquely define H_A, the gradient of the function $H = H(A)$ with respect to $\langle\ ,\ \rangle$, via

(2.5) $$X(H) = \langle X,H_A \rangle, \quad H_A \in \mathcal{L} ,$$

where X is a vector field on $\mathcal{L}^*$, and hence also identifable as an element of $\mathcal{L}^*$. Then since

$$\omega([A,-H_A]_+,[A,\ell_2]_+) = \langle [A,\ell_2]_+,H_A \rangle = [A,\ell_2]_+(H) ,$$

we have that Hamiltons equations with $H = H(A)$, associated with the symplectic structure of (2.4) is, from the definition $\omega(X_H,Y) = Y(H)$,

(2.6) $$\dot{A} = X_H = [H_A,A]_+ .$$

We note that the $H \in \{H | [A,H_A]_+ = 0,$ for all $A \in \theta_B\}$ form an algebra, namely the algebra which characterize the invariants, or constants, of the orbit $\theta = \theta_B$. The description of an orbit, θ_B, consists in finding these constants. We also note from (2.4,6), that the Poisson bracket associated with ω, $\{\ ,\ \}$, is

$$\{H^{(1)},H^{(2)}\}(A) \equiv \omega(X_{H^{(1)}},X_{H^{(2)}})(A) = \langle A,[H_A^{(1)},H_A^{(2)}]\rangle \quad . \tag{2.7}$$

We now specialize these considerations, and for that we introduce some notation. Let the shift operator ξ, acting on R^n be defined by

$$(\xi v)_i = v_{i+1},\ v = (v_1,\ldots,v_n)^T \in R^n \ , \tag{2.8}$$

where we define $v_i \equiv 0$ if $i \notin \{1,2,\ldots,n\}$.

In addition, we associate with $a \in R^n$, the multiplication operator $a\cdot$, namely

$$(a \cdot v)_j = a_j v_j, \qquad j = 1,2,\ldots,n \ . \tag{2.9}$$

With these definitions, we now define

$$A_{k,j} = \{A | A = \sum_{k\le i\le j} a^{(i)} \cdot \xi^i,\ a^{(i)} = (a_0^{(i)},a_1^{(i)},\ldots,a_i^{(n-1-i)},0,0,\ldots,0),\ a^{(i)} \in R^n\} \ . \tag{2.10}$$

Note $a_i^{(j)} = A_{i,i+j}$. For example, if $A = \sum_{i\ge 0} a^{(i)} \cdot \xi^i$, $B = \sum_{i\ge 0} \xi^{-i} \cdot (b^{(i)} \cdot)$, then $\langle A,B\rangle = \sum_{i\ge 0} (a_i,b_i)$, with $(\ ,\ )$ the scalar dot product in R_n, $\langle\ ,\ \rangle$, defined by (2.1). In the future, we shall omit the dot in $a \cdot \xi$ when there is no possibility of misunderstanding. Clearly $\mathcal{L} = A_{-n,0}$, $G \subset A_{-n,0}$, and $\mathcal{L}^* = A_{0,n}$. Along with the grading inherent in the specification of the $A_{k,j}$'s, we have the natural projections $P_{k,j}$ into the $A_{k,j}$.

We shall restrict $A \in \mathcal{L}^*$, such that $A \in A_{0,m}$, $0 < m \le n$, and we observe from (2.2) that $B \in A_{0,m}$ implies $\theta_B \subset A_{0,m}$, and also $[H_B,B] \in A_{-n,m}$, which we indicate by replacing the subscript $+$ in (2.6) with a superscript m, i.e.

$$\dot{A} = [H_A,A]^m \ . \tag{2.11}$$

Since from (2.11), $[H_A,A]^m$ only depends on the part of H_A contained in $A_{-m,0}$, we

may as well restrict H to only be a function of the $A_{i,j}$'s such that $0 \leq j - i \leq m$. Note that clearly $A_{0,m}$ is not an orbit, for if $D \subset \theta_B$, $trB = trD$. This is a consequence of $H = trA$, $H_A = I$, implies $[H_A, A]^m = 0$ in (2.11), and so trA is an orbit invariant. It is easy to see that in general there must be more invariants and we shall not discuss this point, but refer the reader to [18], which discusses this problem fully for the special case of orbits of maximum possible dimensionality.

We compute (2.11) in coordinates, using the notational (2.8)-(2.10). We have that

$$(2.12) \qquad A = \sum_{k=0}^{m} a_k \xi^k, \; H_A = \sum_{j=0}^{m} \xi^{-j} \frac{\partial H}{\partial a^{(j)}}, \quad \text{where} \quad \left(\frac{\partial H}{\partial a^{(j)}}\right)_s = \frac{\partial H}{\partial A_{s,s+j}},$$

hence by (2.11),

$$\dot{A} = \sum_{k=0}^{m} \dot{a}^{(k)} \xi^k = [H_A, A]^m = \sum_{\nu=0}^{m} \sum_{j \geq 0} \left[\xi^{-j}\left(\frac{\partial H}{\partial a^{(j)}} \cdot a^{(\nu+j)}\right] - \left[a^{(\nu+j)} \cdot \left(\xi^{\nu} \frac{\partial H}{\partial a^{(j)}}\right)\right] \cdot \xi^{\nu},$$

and so Hamilton's equations are

$$\dot{a}_t^{(\nu)} = \dot{A}_{t,t+\nu} = \sum_{j \geq 0} \left\{\frac{\partial H}{\partial A_{t-j,t}} A_{t-j,\nu+t} - A_{t,t+\nu+j} \frac{\partial H}{\partial A_{t+\nu,t+\nu+j}}\right\}, \; 0 \leq \nu \leq m, \; 0 \leq t \leq n-1-\nu.$$

For example, if we set $m = 1$, $A_{i,i} = b_i$, $A_{j,j+1} = a_j$, then the above expression yields for (2.11),

$$\dot{b}_i = (a_{i-1} H_{a_{i-1}} - a_i H_{a_i}), \quad \dot{a}_i = a_i (H_{b_i} - H_{b_{i+1}}),$$

hence since $\{F,H\} = X_H F$, we have

$$\{F,H\} = \Sigma \, (a_{i-1} H_{a_{i-1}} - a_i H_{a_i}) F_{b_i} + \Sigma \, a_i (H_{b_i} - H_{b_{i+1}}) F_{a_i},$$

the well known Poisson bracket of the Toda system [2]. In this case it is only necessary to impose the condition $\Sigma \, b_i$ = constant, to specify an orbit, assuming none of the a_i's are zero. The property of a_i being zero is an orbit invariant.

We now make an important observation due to P. Moerbeke in an unpublished version of [2], but first some notation. We write every matrix $M = M^+ + M^0 + M^-$, with M^+ the strictly upper triangular part, M^0 the diagonal part, etc., which brings us to:

Theorem 1. (P. Moerbeke). Let L be the real symmetric matrix $A + (A^+)^T$. Then if $H = H_f(A) = tr\, f(L) = \langle f(L) \rangle$, Hamiltonian equations, $\dot{A} = [H_A, A]^m$, imply

that L satisfies the Lax isospectral flow equation

(2.13) $$\dot{L} = [P,L], \quad \text{with} \quad P = P_f = f(L)^+ - f(L)^- .$$

Moreover this implies that $\{H_f,H_g\} = 0$ for all polynomial f, g, i.e. the H_f are in involution with respect to $\{\ ,\ \}$.

Proof. We give a 'functional' version of P. Moerbeke's proof, the unpublished version of [2]. In the course of the proof we shall show the necessary fact $[P,L] \in A_{-m,m}$.

We compute, using the notation of (2.8)-(2.10),

$$\frac{\partial H}{\partial a^{(i)}} = \frac{\partial \langle f(L)\rangle}{\partial a^{(i)}} = \langle f'(L), \frac{\partial L}{\partial a^{(i)}}\rangle = \begin{cases} \langle f'(L), \xi^{i} + \xi^{-i}\rangle, & 1 \le i \le m, \\ \langle f'(L), \xi^{0}\rangle, & i = 0 \end{cases} .$$

Hence if $f'(L) = \sum_{i=0}^{n} (f'(L))^{(i)}\xi^{i} + \sum_{i=-n}^{-1} \xi^{-i}\cdot(f'(L))^{(i)}$, $\frac{\partial H}{\partial a^{(i)}} = 2(f'(L))^{(i)} - \delta_{i,0}(f'(L))^{(i)}$, $i = 0,1,\ldots,m$, and thus

(2.14) $$H_A = [2f'(L)^- + f'(L)^0]_{-m} ,$$

where $[\ ]_{-m}$ denotes the natural projection into $A_{-m,0}$.

We now use the fundamental observation $[f'(L),L] = 0$, observing $0 = [f'(L),L] = [f'(L)^+ + f'(L)^0 + f'(L)^-,L]$, hence

$$[P,L] = [f'(L)^+ - f'(L)^-,L] = [L,f'(L)^0 + 2f'(L)^-] ,$$

but

$$[L,f'(L)^0 + 2f'(L)^-] = [A + L^-, f'(L)^0 + 2f'(L)^-] ,$$

and so $[P,L]_+ = [A,f'(L)^0 + 2f'(L)^-]_+ = [A,f'(L)^0 + [2f'(L)]_{-m}]^m$, (and thus by the skew-symmetry of P, $[P,L] \in A_{-m,m}$), and by (2.14)) $= [A,H_A]^m$.

We thus have shown

$$\dot{A} = [A,H_A]^m \quad \text{implies} \quad \dot{L} = [P,L], \quad H = H_f .$$

Since $\dot{L} = [P,L]$, $P = P_f$ when $H = H_f$,

$$\frac{d}{dt} H_g = \frac{d}{dt} \langle g(L)\rangle = \langle g'(L) \cdot \dot{L}\rangle = \langle g'(L),[P,L]\rangle = \langle P,[L,g'(L)]\rangle = 0 ,$$

since $[L,g'(L)] = 0$, but $\frac{d}{dt} H_g = \{H_g, H_f\}$, and so we have proven the statement concerning the involution of the H_f's. This concludes the proof of Theorem 1. We refer the reader to theorems analogous to Theorem 1, for other finite systems, in [9], [10].

There is more information, namely recursion relations, or so-called Lenard relations to be gleaned from the companion identity to the fundamental relation $[f'(L),L] = 0$, namely the stronger statement, $L^{j+k} = L^j \cdot L^k$, which we shall discuss in Section 4, but for the moment we shall go on in the next section to the differential operator case. In preparation for the next section we mention immediate generalizations of the above discussion.

Remark 1. One generalization is that one may consider the $a^{(i)}$ in $A = \sum_{0 \le i \le m} a^{(i)} \xi^i$ to be n-periodic, i.e. ξ acts on R_n^∞, where $v \in R_n^\infty$ means $v = (\ldots, v_{-1}, v_0, v_1, \ldots)^T$, and v is an n-periodic vector, i.e. $v_{j+n} = v_j$ for all j, and $a^{(i)} \in R_n^\infty$. As before, A, L, may be represented by an $n \times n$ matrix, and all the proofs given above apply to this case, with slight modifications, provided we don't think in terms of matrices, but operators, for instance $A = \sum_{i=0}^{m} a^{(i)} \xi^i$, where we require that $m \le [\frac{n}{2}]$. An equally interesting generalization is when ξ acts on R^∞, where $v \in R^\infty$ means $v = (\ldots, v_{-1}, v_0, v_1, \ldots)^T$, and $A = \sum_{i=0}^{m} a^{(i)} \xi^i$, $a^{(i)} \in R^\infty$. In the later case the (formal) Lie group and Lie algebra are respectively

$$G = \{ \sum_{i \le 0} a^{(i)} \xi^i \mid a^{(i)} \in R^\infty,\ a_j^{(0)} > 0 \text{ for all } j\},\quad \mathcal{L} = \{ \sum_{i \le 0} a^{(i)} \xi^i \mid a^{(i)} \in R^\infty\},$$

and we have for the dual of $\mathcal{L}$,

$$\mathcal{L}^* = \{ \sum_{i=0}^{N} a^{(i)} \xi^i \mid a^{(i)} \in R^\infty,\ N < \infty\}.$$

We finally motivate the constructions of this section, and especially the next section, by a brief discussion of Fourier theory on finitely generated abelian groups. For a more complete discussion, we refer the reader to [16]. Since such a group can be decomposed as a direct sum of copies of Z, Z_n, where Z is the integers, Z_n the integers $\{0,1,\ldots,n-1\}$ with modulo n addition, it's sufficient to discuss Z, Z_n.

We first discuss Z_n, with $g \in Z_n$, i.e. $g \in \{0,1,\ldots,n-1\}$. The character group of $Z_n, \hat{Z}_n$ are the functions χ, from Z_n into the complex numbers, which multiply by pointwise multiplication, having the properties

$$\chi(g_1) \cdot \chi(g_2) = \chi(g_1 + g_2), \quad |\chi(g)| = 1 ,$$

note $\chi(-g) = \chi(g)^*$, with $*$ being complex conjugation, and $\chi(1) = 1$. Clearly the character group $\hat{Z}_n = \{\chi = \chi_k | \chi_k(j) = a_k^j,\ a_k = e^{\frac{2\pi i}{n}k},\ k \in \{0,1,\ldots,n-1\}$, and thus $Z_n, \hat{Z}_n$ are identical as abstract groups. Note we may change our viewpoint and think of $\chi_{(\cdot)}(j)$ *as a character of* $\hat{Z}_n$. If we define the inner product on functions of Z_n, $(f_1,f_2)_{Z_n} = \sum_{g \in Z_n} f_1(g) f_2^*(g)$, then clearly $(\chi_j(\cdot),\chi_k(\cdot)) = n\delta_{jk}$, and viewing $\chi_{(\cdot)}(j)$ as characters in $\hat{Z}_n$ we have by the above duality $(\chi_{(\cdot)}(j),\chi_{(\cdot)}(k))_{\hat{Z}_n} = n\delta_{jk}$. These two identities immediately yield the Fourier decomposition of functions on Z_n and the Plancheral identity, namely if f is a function on Z_n, we have

$$f = \sum_{j \in Z_n} \hat{f}(j)\chi_j, \quad \hat{f}(j) = n^{-1}(f,\chi_j), \quad \text{i.e. } \hat{f}(\cdot) \text{ is a function on } \hat{Z}_n ,$$

$$\|f\|^2_{Z_n} = (f,f)_{Z_n} = \sum_{i \in Z_n} |f(i)|^2 = n\|\hat{f}\|^2_{\hat{Z}_n} = n(\hat{f},\hat{f})_{\hat{Z}_n} = n \sum_{j \in \hat{Z}_n} |\hat{f}(j)|^2 .$$

If we identify the functions $f \in F(Z_n)$ with n periodic vectors $\tilde{f}$ in R_n^∞, by the recipe $(\tilde{f})_j = f(j)$, then we may define the shift operator $\xi : F(Z_n) \mapsto F(Z_n)$, through $[\xi(\tilde{f})]_j = \tilde{f}(j+1)]$. Clearly the characters of Z_n are a complete list of the eigenfunctions of ξ which have the value 1 at $j = 0$, i.e. $\xi(\chi_j) = a_j\chi_j$, from which we conclude if we decompose an operator B acting on $F(Z_n) \cong R_n^\infty$ into $B = \sum_{j \in Z_n} b^{(j)} \cdot \xi^j = P(\xi)$, where $\cdot$ denotes pointwise multiplication, that

$$B(\chi_k) = \sum_{j \in Z_n} a_k^j b^{(j)} \chi_k = \{\sum_{j \in Z_n} a_k^j b^{(j)}\} \cdot (\chi_k) = P(a_k) \cdot \chi_k, \quad \text{and so}$$

$$B(f) = B(\sum_{j \in \hat{Z}_n} \hat{f}(j)\chi_j) = \sum_{j \in \hat{Z}_n} \hat{f}(j) \cdot \{\sum_k a_j^k b^k\} \cdot \chi_j\} = \sum_{j \in \hat{Z}_n} \hat{f}(j) P(a_j) \cdot \chi_j ,$$

hence we have

$$(2.15)\qquad B(\chi_k) = P(a_k)\cdot\chi_k,\quad B(f) = \sum_{j\in\hat{Z}_n} \hat{f}(j)P(a_j)\cdot\chi_j,\quad \text{if}\quad B = P(\xi)\ .$$

The above situation is to be compared with the Fourier theory of S^1. Here $(S^1)^\wedge = \{e^{2\pi inx} = \varphi_n(x)\,|\,,\ n\in Z\}$, and thus as an abstract group $(S^1)^\wedge \cong Z$. The Fourier decomposition theorem and Plancharel identity takes the form for $f\in L^2(S^1)$,

$$f = \sum_{n\in Z} \hat{f}(n)\varphi_n(x),\qquad \hat{f}(n) = (f,\varphi_n) = \int_0^1 f\bar{\varphi}_n(x)dx\ ,$$

i.e. $\hat{f}(\cdot)$ is a function on $Z = (S^1)^\wedge$ and

$$(2.16)\qquad ||f||^2_{S^1} = (f,f) = \int_0^1 |f|^2dx = \sum_{n\in Z} |\hat{f}(n)|^2 = (\hat{f},\hat{f})_Z\ ,$$

which moreover implies $\hat{f}\in L^2(Z)$.

In addition, since $\frac{-iD}{2\pi}(\varphi_n) = n\varphi_n$, we have if $L = \Sigma\, u_j(\frac{-iD}{2\pi})^j = P(\frac{-iD}{2\pi})$, that $L\varphi_n = P(n)\cdot\varphi_n$, $L(f) = \Sigma\,\hat{f}(n)P(n)\varphi_n(x)$, and thus by the previous discussion we shall think of the shift operator ξ, and $(\frac{-iD}{2\pi})$ as analogous quantities in the two Fourier theories.

The discussion of the Fourier theory of Z has already taken place, for from the above we have $(S^1)^\wedge = Z$, and conversely $Z^\wedge = (S^1)^\wedge = S^1$, for as before, we may interpret the characters $\varphi_n(\cdot)$ of S^1, as characters $\varphi_{(\cdot)}(x)$ of Z, i.e. functions of Z. Thus if we read the Fourier series decomposition and Plancharel identity for S^1 backwards in (2.16), we have the corresponding decomposition and identity theorems for $L^2(Z)$. The same remarks concerning the analogy between ξ, $\frac{-iD}{2\pi}$ apply, for if $P(\xi) = \sum_{j\in Z} a^j\cdot\xi^j$, then

$$P(\xi)\bar{\varphi}_n = P(a_x)\bar{\varphi}_n,\quad a_x = e^{-2\pi ix},\quad P(\xi)\hat{f}(n) = \int_0^1 f\ P(a_x)\bar{\varphi}_n(x)dx\ ,$$

which is in analogy to the corresponding formulas (2.15) for both Z_n involving ξ, and S^1 involving $\frac{-iD}{2\pi}$. We remark that its suggestive to "identity" $\frac{-iD}{2}$ and $\frac{1}{2}(\xi - \xi^{-1})$, but that identification isn't the correct one for our purposes. We also

remark that the only definition of trace for an operator $P(\xi) = \sum_{j\in Z} a^j \cdot \xi^j$, is $\operatorname{tr} P = \sum_{j\in Z, Z_n} (a^0)_j$, depending on which case we are in. This is of course consistent with the matrix terminology used in this section. An important question answered in the next section is the corresponding quantity for formal pseudo-differential operators.

Remark 2. We note that formula (2.7) in fact defines a natural Poisson bracket for $\mathcal{L}^*$, as is well known. This is easily seen by using the symmetry of $\langle\ ,\ \rangle$, the symmetry of the Hessian H_{AA}, and the Jacobi identify for the bracket $[\ ,\ ]$.

However, given a Poisson bracket, $\{\ ,\ \}$, i.e. a skew symmetric bilinear derivation satisfying the Jacobi identity on a C^∞ manifold M, one automatically has a 'stratification' of M by symplectic leaves, i.e. a 'symplectic stratification'. By stratification of M, we mean that at every point $p \in M$, there exists a smooth leaf Σ_p (possibly zero dimensional) running through p, Σ_p, and moreover Σ_p is part of a foilation near p which foilates nearby Σ_s's, perhaps trivially, and in addition M is the direct union of these leaves. By symplectic stratification, we mean that the leafs are assumed to be C^∞ symplectic manifolds, (or points), with the symplectic two-form $\omega = \omega(p)$ on Σ_p also varying smoothly across the leafs. In other words, for all p, given any smooth vector fields X, Y defined on a neighborhood of p on N_p, such that $X(s), Y(s) \subset T\,\Sigma_s(s)$, $s \in N_p$ (such vector fields necessarily exist), $\omega(X(s),Y(s))(s)$ is smooth in s.

To show that we have the above 'symplectic stratification', first observe that since $\{f,g\}$ is a derivation in its arguments, it is local, i.e. its value of p just depends on the first derivatives of its arguments at p, as is seen by substituting into $\{\cdot,f\}$ a function which vanishes quadratically fast at the point p, and using the product rule of derivations. Next define the Hamiltonian vector field $X_f(g) = \{g,f\}$, and since we may restate the Jacobi identity for $\{\ ,\ \}$ as $[X_{f_1}, X_{f_2}] = -X_{\{f_1,f_2\}}$, we see that $\Lambda = \bigcup_{p\in M} \Lambda_p$, $\Lambda_p = \operatorname{span}(X_{f(p)})_{f\in C^\infty(M)}$, is in fact an integrable distribution in the sense of Frobenius. By this we mean that near p, one has an integrable

distribution Ω_s, $s \in N_p$, such that $\Omega_p = \Lambda_p$, $\Omega_s \subset \Sigma_s$, and dimension (Ω_s) = dimension $(\Lambda_p) = n_p$. This is clearly the case, for if $X_{f_1}(p), X_{f_2}(p), \ldots, X_{f_{n_p}}(p)$ span Λ_p, just define $\Omega_s = \mathrm{span}(X_{f_1(s)}, \ldots, X_{f_{n_p}(s)})$, $s \in N_p$. By the formula $d\phi X_f \phi^{-1} = X_{f \circ \phi^{-1}}$, where $\phi = \phi^{t_1} \circ \phi^{t_2}, \ldots \circ \phi^{t_n}$, ϕ^{t_i} the flow generated by a vector field of the form X_g, we see that a leaf cannot change its dimensionality, which proves the assertion that M equals the direct sum of the leaves. We call the leaf running through p, Σ_p. We define the two form on Σ_p at p by $\omega(Z,Y) = \{f,g\}(p)$, where $Z = X_f(p)$, $Y = X_g(p)$. Clearly ω is well defined. To see $d\omega = 0$, just apply (using the intrinsic definition of d), $X_{f_1}(p)$, $X_{f_2}(p)$, $X_{f_3}(p)$ to $d\omega$, and use the Jacobi identity coupled with $d^2 f = 0$. That ω is nondegenerate is automatic, since $T\Sigma_p(p) = \Lambda_p$. This proves our assertion. Note that given a 'symplectic stratification' of M, one automatically has a Poisson bracket, for simply define $\{f,g\}(p) = \{f|_{\Lambda_p}, g|_{\Lambda_p}\}|_{\Lambda_p}$, where $f|_{\Lambda_p} = f$ restricted to Λ_p, etc., and $\{\ ,\ \}_{\Lambda_p}$ denotes the Poisson bracket Λ_p inherits as a symplectic manifold. That $\{\ ,\ \}$ is a Poisson bracket follows from $\{\ ,\ \}_{\Lambda_p}$ being one, and the stipulation placed upon the smoothness with which the symplectic two form on the leaves may vary.

We also note that if we assume, for instance, that M is not compact and $X_f \equiv 0$ implies $M \not\in C_0^\infty(M)$, the functions of compact support, then if G is the group with Lie algebra $\Delta = \{X_f | f \in C_0^\infty(M)\}$, we may think of $G \times M \subset T^*G$, via the pairing $\langle p, X_f \rangle = f(p)$, and the above construction as an example of the Kostant-Kirillov orbit construction. This will be discussed elsewhere.

Thus given a Poisson bracket, it is natural to compute the symplectic leaves it induces. In the case of (2.7), the leaves are nothing more than the orbits of Kostant-Kirillov. It will, however, not always be the case that a finite dimensional group generates the orbit structure on M.

3. The Generalized Korteweg-deVries Equations

In order to apply the considerations discussed in Section 2 to the generalized Korteweg-deVries equation, we need to define the appropriate formal Lie group G. To this end we introduce the commutative ring $R = R(..,a_{-i},..,a_0,..,a_j,..) = R(a)$ over the complex numbers equipped with a natural derivation D, consisting of polynomials in the a_i and their derivatives. One defines the 'indefinite integrals', $I = R/DR$, i.e. R modulo DR, and we shall use $\doteq$ for the equality sign in I. Note that D treats I elements as if they were constants.

We now define the ring of formal asymptotic pseudo-differential operators over R, Φ, to be the ring of formal Laurant series in the variable ξ, where we may heuristically think of ξ as contained in a real neighbourhood of ∞ which is moreover embedded in a complex neighbourhood of ∞, and we shall take the coefficients of the series to be R elements, and hence D extends to Φ naturally. We shall also require that each series terminates at some ξ^j, $j < \infty$. The ring multiplication shall be just the one used in pseudo-differential operator theory, namely

$$\phi_1 \circ \phi_2 = \sum_{\nu \geq 0} (\partial_\xi)^\nu \phi_1 \cdot (-iD)^\nu \phi_2 \; . \tag{3.1}$$

Note we may write

$$\Phi = \{\phi = \sum_{-\infty < i \leq N < \infty} a_i \xi^i \,|\, a_i \text{ a generator of } R(a)\}. \tag{3.2}$$

The motivation for these definitions was discussed in the introduction, and we reiterate that we heuristically think of Φ elements as germs at ∞. We briefly discuss (3.1). If R elements were just C^∞ functions of x, $D = \frac{d}{dx}$, then the differential operator $P(-iD)$, with P a polynomial, acts on $e^{ix\xi}$ via $P(-iD)(e^{ix\xi}) = P(\xi) \cdot e^{ix\xi}$, and clearly $[P_1(-iD)(P_2(-iD)(e^{ix\xi}))] = (P_1(\xi) \circ P_2(\xi)) \cdot e^{ix\xi}$, with $\circ$ defined in (3.1). Thus Φ models the image of the algebraic isomorphism implicit in the above comments, $P(-iD) \mapsto P(\xi)$, in a formal way, and moreover extends the image of the isomorphism. These comments also explain why $\circ$ is associative.

The care taken in the definition of Φ was in order to define the trace functional $\langle\ \rangle : \Phi \mapsto I$, by, assuming $\phi = \Sigma a_i \xi^i$,

(3.3) $$\text{tr}\ \phi \equiv \langle \phi \rangle \doteq \bar{a}_{-1} \in I,$$

where if $\phi \in R$, $\bar{\phi} \in I$ denotes the element in I with representative element ϕ. We now state the simple but fundamental result concerning $\langle\ \rangle$.

Theorem 2. If $[\phi_1,\phi_2] = \phi_1 \circ \phi_2 - \phi_2 \circ \phi_1$, we have

(3.4) $$\langle [\phi_1,\phi_2] \rangle = 0 .$$

Proof: It's here that the duality between x,ξ, in Fourier theory plays a crucial role, as evidenced by its appearance in the role of multiplication (3.1). Its immediate from (3.1) that

$$[\phi_1,\phi_2] = \frac{\partial}{\partial \xi} \alpha + D\beta, \quad \alpha,\beta \in \Phi ,$$

hence projecting into $I = R/DR$ via $\bar{\ }$, we have

$$\overline{[\phi_1,\phi_2]} \doteq \frac{\partial \bar{\alpha}}{\partial \xi} ,$$

and so clearly $\overline{[\phi_1,\phi_2]}$ can have no ξ^{-1} term, which, by (3.3), concludes the proof of Theorem 2.

We also remark that one can prove Theorem 2 by direct computation, which is useful for Section 4 but then one fails to see why our choice for $\langle\ \rangle$ is really unique.

As a consequence of (3.4) we make the important definition, analogous to (2.1) of $\langle\ ,\ \rangle$,

(3.5) $$\langle \phi_1,\phi_2 \rangle \equiv \langle \phi_1 \circ \phi_2 \rangle ,$$

and so by Theorem 2, we have that $\langle\ ,\ \rangle$ is symmetric in its arguments, which is the crucial point in order that the operation $[\phi,\cdot]$ be skew-symmetric with respect to $\langle\ ,\ \rangle$, which played an important role in the considerations of Section 2.

We now define Φ_λ to be the ring over Φ of formal Laurant series in λ, which we shall heuristically think of as being a member of the complex numbers in a deleted neighbourhood of ∞, minus the real numbers. Φ_λ inherits its multiplication structure

from Φ and the law of exponents. We now single out Φ_n, a submanifold of Φ, defined by

(3.6) $$\Phi_n = \{\phi = \xi^n + \sum_{0\le i\le n-2} a_i\xi^i \mid a_i \text{ a generator of } R(a)\} .$$

We can now define the resolvent operator $\mathcal{R}_n$,

$$\mathcal{R}_n : \Phi_n \mapsto \Phi_\tau, \quad \tau = \lambda - \xi^n, \quad \mathcal{R}_n(\phi) \equiv \phi_\lambda ,$$

by requiring

(3.7) $$\phi_\lambda \circ (\lambda - \phi) = (\lambda - \phi) \circ \phi_\lambda = 1 .$$

We compute ϕ_λ in the following way, writing $\phi = \xi^n + \hat{\phi}$,

$$\phi_\lambda \circ (\lambda - \phi) = \phi_\lambda \circ \{(\lambda - \xi^n) - \hat{\phi}\} = \phi_\lambda \cdot (\lambda - \xi^n) - \phi_\lambda \circ \hat{\phi} = 1 ,$$

hence

$$\phi_\lambda - \frac{\phi_\lambda \circ \hat{\phi}}{\lambda - \xi^n} = \frac{1}{\lambda - \xi^n} ,$$

and so ϕ_λ satisfies an equation of the form $(I - T)x = y$, with $x = \phi_\lambda$, $y = (\lambda - \xi^n)^{-1}$, $T(x) = \dfrac{x \circ \hat{\phi}}{(\lambda - \xi^n)}$, hence $\phi_\lambda = \sum_{j\ge 0} T^j(\lambda - \xi^n)^{-1}$, from which it's easy to see by (3.1) that

(3.8) $$\phi_\lambda = \Sigma\, \phi_{m,\ell}(-1)^{\frac{m}{n}}\xi^m(\xi^n - \lambda)^{-1-\frac{m+\ell}{n}}, \quad m \ge 0,\ \ell \ge 0,\ \frac{m+\ell}{n} \text{ an integer} ,$$

for $\phi_{m,\ell} \in R_n = R_n(a_0, a_1, \ldots, a_{n-2})$, where R_n is the ring whose elements are finite sums and products of a_i, $0 \le i \le n-2$, (see (3.6)), and their derivatives. In (3.8) we use the notation of Gelfand-Dikii [1]. Using the construction of (3.8), we can mimic the construction of R. Seeley [7], in a formal way, namely we define for 'nice' f,

(3.9) $$f(\phi) \equiv [f(\lambda)\phi_\lambda]_{(\lambda,\xi^n)} ,$$

where for $h = h(\lambda)$, $[h]_{(\lambda,\xi^n)}$ is the formal residue term of h at ξ^n, i.e. the coefficient of the $(\lambda - \xi^n)^{-1}$ term of the Laurent series in $(\lambda - \xi^n)$, given by formally expanding $h = h(\lambda)$ about ξ^n. We keep this notation. This of course necessitates expanding $f(x)$ about $x = \xi^n$, which places restrictions on the choice

of f. One uses (3.8) to compute (3.9). In practice one may have to extend $\Phi \mapsto \tilde{\Phi}$ to include $f(\phi)$. As a simple example, by (3.8), (3.9), one easily computes

$$\phi^s = \Sigma\, \phi_{m,\ell} \binom{s}{\frac{m+\ell}{n}} \xi^m (\xi^n)^{\left(s - \frac{\ell+m}{n}\right)} = \sum_{\ell \geq 0} \Delta_{\ell, s\cdot n} (\xi^n)^s \xi^{-\ell} \in \tilde{\Phi} ,$$

where $\Delta_{\ell,s\cdot n} \equiv \sum_m \phi_{(m,\ell)} \binom{s}{\frac{m+\ell}{n}}$. For the case that $s = \frac{N}{n}$, $N \in Z$, but not a multiple of n, we have

$$\text{(3.10)} \qquad \phi^{\frac{N}{n}} = \sigma \cdot \sum_{\ell \geq 0} \Delta_{\ell,N} \xi^{N-\ell}, \quad \text{hence} \quad \langle \phi^{\frac{N}{n}} \rangle = \bar{\Delta}_{N+1,N} \cdot \sigma ,$$

where we may heuristically interpret $\sigma = \left\{ \begin{matrix} \text{sign}(\xi), & n\text{-odd} \\ 1 & , n\text{-even} \end{matrix} \right\}$, but it is really just an adjoined symbol obeying the formal rule of multiplication $\sigma^2 = 1$.

We amplify (3.10). Here we think of $\phi^{\frac{N}{n}} \in \hat{\Phi}, \hat{\Phi}$ and extension of Φ, $\hat{\Phi} = \Phi \oplus \sigma \cdot \Phi$. We must think of σ as just an adjoined symbol, obeying $\sigma^2 = 1$, but if we interpret $\phi \in \hat{\Phi}$ as a series in real ξ embedded in a neighbourhood of ∞ in the complex plane, then σ has the interpretation given above. Implicitly we have modified the ring R, i.e. $R \to R \oplus \sigma \cdot R$, hence $I \to I \oplus \sigma \cdot I$. That the trace has all of its usual properties follows from $\frac{\partial}{\partial \xi} \sigma = 0$. In general $\phi^s \in \sum_{\tau, s \in C} \oplus (\sigma_\tau \cdot \Phi \cdot \xi^s)$, with $\sigma_{\tau_1} \cdot \sigma_{\tau_2} = \sigma_{\tau_1 + \tau_2}$, ($C$ the complex numbers). Hence it's obvious that if s is irrational, ϕ^s has no residue term and so $\langle \phi^s \rangle = 0$, $s \neq \frac{N}{n}$, $N \notin Z$. In the future it shall be understood we are working with the enlarged ring, $R \oplus \sigma \cdot R$, but we shall not change notation.

In addition, we define $K(\phi,\lambda)$, using the notation of (3.9), and (3.8), as

$$\text{(3.11)} \qquad K(\phi,\lambda) \doteq [\phi_\lambda]_{(\xi,\lambda^{1/n})} \doteq \sum_{\ell \geq 0} \mu_\ell (\lambda^{\frac{1}{n}})^{-(n+\ell-1)} \in \bar{\Phi}_{(\lambda^{1/n})}, \quad \mu_\ell \in I .$$

This is easily seen by computing the formal residue of terms of the form $\tau_{m,n} = \xi^m (\xi^n - \lambda)^{-1-\frac{m+\ell}{n}}$ in (3.8), which equivalently comes down to expanding $\tau_{m,n}(\xi)$

in powers of $(\xi - \lambda^{1/n})$ and picking off the $(\xi - \lambda^{1/n})^{-1}$ coefficient; in other words, we think in terms of expanding $\varphi_\lambda = \phi_\lambda(\xi)$ about $\lambda^{1/n}$. Notice we have formally declared $(\lambda^{1/n})^n = \lambda$ in $\tau_{m,n}$ rather than "pick a branch". In analogy to Section 2, we define $A_{i,j} = \{\sum_{s=i}^{j} \phi_s \xi^s | \phi_s \in R\}$, and the natural projections $P_{i,j}$ into the $A_{i,j}$'s. We shall work with the formal Lie group $G = 1 + A_{-\infty,-1}$, with formal Lie algebra $\mathcal{L} = A_{-\infty,-1}$. Alternately we shall choose to represent $\mathcal{L}$ in the following form

$$\mathcal{L} = \{\sum_{j\geq 0} (\xi - iD)^{-j-1}\phi_j | \phi_j \in R\}\ , \tag{3.12}$$

where $(\xi - iD)^{-j-1}\phi \equiv \sum_{v\geq 0} \xi^{-j-1-v}\binom{j+v}{v}(iD)^v\phi$, $j \geq 0$. It is obvious that the two definitions of $\mathcal{L}$, are the same, in fact if

$$\begin{aligned}\sum_{k\geq 0} c_k\xi^{-k-1} &= \sum_{j\geq 0} (\xi - iD)^{-j-1}b_j = \sum_{\substack{v\geq 0\\ j\geq 0}} \binom{j+v}{v}(iD)^v b_j \xi^{-1-j-v} \\ &= \sum_{k\geq 0} (\sum_{v=0}^{k} \binom{k}{v}(iD)^v b_{k-v})\xi^{-k-1}\ ,\end{aligned} \tag{3.13}$$

we must have

$$c_k = \sum_{v=0}^{k} \binom{k}{v}(iD)^v b_{k-v}, \quad \text{and so} \quad b_k = \sum_{k=0}^{k} \binom{k}{v}(-iD)^v c_{k-v}\ . \tag{3.14}$$

The second relation in (3.14) being an easy consequence of the first. We shall denote by $\tilde{A}_{k,j} = \{\sum_{s=k}^{j} (\xi - iD)^s \cdot \phi_s | \phi_s \in R\}$, and by $\tilde{P}_{k,j}$ the natural projections into $\tilde{A}_{k,j}$. We identify the dual of $\mathcal{L}$,

$$\mathcal{L}^* = \{\sum_{\infty > N \geq i \geq 0} \phi_i \xi^i | \phi_i \in R\} = A_{0,\infty}\ . \tag{3.15}$$

If $A = \sum_{i\geq 0} a_i\xi^i \in \mathcal{L}^*$, $B = \sum_{j\geq 0} (\xi - iD)^{-j-1}b_j$, then

$$\langle A,B\rangle \doteq \langle A\circ B\rangle \doteq \sum_{i\geq 0} \overline{a_i b_i}\ , \tag{3.16}$$

as easily follows from (3.15), (3.1). Note this is in complete analogy with the formula

following (2.10). In other words, $(\xi - iD)^{-j-1}$ is due to ξ^j, $j \geq 0$ under $\langle\ ,\ \rangle$, i.e.

(3.17) $$(\xi^j)^* = [(\xi - iD)^{-j-1}],\quad [(\xi - iD)^{-j-1}]^* = \xi^j, \quad j \geq 0 ;$$

where * denotes duality under $\langle\ ,\ \rangle$. Heuristically 'total integration' $j + 1$ times is dual to 'differentiation' j times. We now are in a position to state the main theorem of this paper, but we first mention that we shall think of covariant tensors as multilinear operators on vector fields, linear over I.

Theorem 3. If $A \in \theta_B = \{[g^{-1}Bg]_+ | g \in G\}$, where + denotes projection onto $A_{0,\infty}$, and $B \in A_{0,m}$ i.e. θ_B is just the orbit of B under the dual of the adjoint action, and so $A = \sum_{i=0}^{m} a_i \xi^i$, then the (formal) natural symplectic 2-form ω on θ_B is given by, at the point A, with $\ell_i \in \mathcal{L}$,

$$\omega([A,\ell_1]_+,[A,\ell_2]_+) \doteq \langle A,[\ell_1,\ell_2]\rangle \doteq \langle [A,\ell_1]_+,\ell_2\rangle .$$

Thus Hamilton's equations associated with ω, for the Hamiltonian $H \doteq H(A) \in I$, a polynomial in a_i, $i = 0,1,\ldots,m$ and its derivatives, is given by

(3.18) $$\dot{A} = X_H = [H_A,A]^{m-2} ,$$

$[\ ]^n$ denoting the natural projection onto $A_{0,n}$, where

(3.19) $$H_A = \sum_{j=0}^{m} (\xi - iD)^{-j-1} \frac{DH}{DA_j} ,$$

with $\frac{DH}{DA_j}$ being the formal variational derivative of $H \in \mathcal{L}$ with respect to A_j. In addition, the (formal) Poisson bracket $\{\ ,\ \}$, is given by

(3.20) $$\{H,F\} \doteq \langle A,[H_A,F_A]\rangle .$$

Finally this Poisson structure agrees with the structure of Gelfand-Dikii [1] if we take $a_m = 1$, $a_{m-1} = 0$. We can do this as a_m, a_{m-1} are orbit invariants. At this point we may restrict our differential ring R to only consist of the polynomials in the a_i's, $i = 0,1,\ldots,m$, and their derivatives, over the complex numbers, with the adjointed branch symbol σ, i.e. $R = R(a_0,a_1,\ldots,a_m)$.

Proof: We are formally in the same situation as in Section 2, and so we shall not repeat the discussion now but merely amplify certain points, and hence just sketch

the proof. For instance if $H \doteq H(A) \in I$, H_A the gradient of H with respect to $\langle \, , \, \rangle$ is first defined uniquely by requiring

$$X(H) \doteq \langle H_A, X\rangle, \quad H_A \in \mathcal{L} ,$$

where $X = \sum_{i\geq 0} X_i \frac{D}{Da_i}$, is a vector field on $\mathcal{L}^*$, which in the inner product is naturally identified with the $\mathcal{L}^*$ element $\sum_{i\geq 0} X_i \xi^i$. For H depending only on $a_0, a_1, \ldots, a_m$, we automatically have $H_A \in A_{-(m+1),-1}$, and then formula (3.19) follows from (3.16). Also see [4] for amplification on related matters. The formulas for θ_B, ω, Hamilton's equations, $\{ \, , \, \}$, were clearly taken over directly from the discussion of Section 2.

We emphasize that these equations can only be taken in a formal sense, which implies that ω is formally closed, (using the intrinsic definition of $d\omega$) and therefore that $\{ \, , \, \}$ is a Poisson bracket, i.e. is a skew symmetric derivation which satisfies the Jacobi identity. We amplify this point, taking the point of view of the formal calculus of variation of Gelfand-Dikii [see 4]. We define the operation of exterior differentiation, d, using the intrinsic definition which relies on the Lie bracket. This definition gives us no trouble, as our ring $R(a)$ is composed of polynomials. For instance for $H \in I$, $dH = \Sigma \frac{DH}{Da_j} da_j$. From the formula for ω given above, one verifies directly that $d\omega = 0$, as indeed is true in the case of a bonified Lie group. In the usual verification that $\{H,F\} \equiv \omega(X_H, X_F)$ satisfies the Jacobi identity, one used that $d\omega = 0$, and that $d\omega(H_H, \cdot) = d(dH) = d^2H = 0$. That $d^2 = 0$ follows from the intrinsic definition of d, and says in this case, in coordinates, that the 2-form

$$d(dH)(\cdot,\cdot) = d(\Sigma \frac{DH}{Da_i} da_i) = \sum_{i<j} \left[(\cdot) \frac{\delta}{\delta a_i}\left(\frac{DH}{Da_j}\right)(\cdot) - (\cdot) \frac{\delta}{\delta a_j}\left(\frac{DH}{Da_i}\right)(\cdot)\right] da_i \wedge da_j = 0 ,$$

where $\frac{\delta}{\delta a_i}\phi$, $\phi \in R$, is the (formal) directional derivative of ϕ with respect to a_i, or equivalently $d^2H(Z,Y) = \langle Z, \delta H_A(Y)\rangle - \langle Y, \delta H_A(Z)\rangle = 0)$, $\delta = \frac{\delta}{\delta A}$, hence

$$\frac{\delta}{\delta a_i} \frac{DH}{Da_j}(\cdot) = \frac{\delta}{\delta a_j} \frac{DH}{Da_i}(\cdot), \quad \text{all } i,j .$$

This is just the well known fact that the directional derivative of H_A is a symmetric operator with respect to $\langle\ ,\ \rangle$. See [4,3] for amplification on this particular case of the Poincaré Lemma, where this fact is discussed and proved differently. One can also show directly from (3.20), that $\{\ ,\ \}$ is a Poisson bracket. One uses for this the symmetry of $\langle\ ,\ \rangle$, the Jacobi identity for $[\ ,\ ]$, and that the formal Gateaux derivative of H_A is a symmetric operator with respect to $\langle\ ,\ \rangle$, which is precisely what one would use in Section 2, as was mentioned in Remark 2.

Finally the remark concerning the Poisson brackets of Gelfand-Dikii follows first from the fact that since a_m, a_{m-1} are orbit invariants they may be taken equal to 1, 0 respectively. Note that in general the symplectic structure will depend on a_m, a_{m-1}, as indeed the following computation will make clear. We now compute (3.18) using formula (3.1), which will finish the proof of Theorem 3. First observe, setting $\frac{DH}{Da_j} = H_{a_j}$

$$H_A = \sum_{j\geq 0} (\xi - iD)^{-j-1} H_{a_j} = \sum_{\substack{j\geq 0\\ s\geq 0}} \xi^{-j-1-s} \binom{j+s}{s} (iD)^s H_{a_j} ,$$

hence

$$\dot{A} = [H_A, A]^{m-2} = \left\{\begin{matrix} (\Sigma\, \xi^{-j-1-s}\binom{j+s}{s} (iD)^s H_{a_j}) \circ (\Sigma\, a_k \xi^k) \\ -(\Sigma\, a_k \xi^k) \circ (\Sigma\, \xi^{-j-1-s}\binom{j+s}{s} (iD)^s H_{a_j} \end{matrix}\right\}^{m-2}$$

$$= \sum_{r=0}^{m-2} \begin{bmatrix} \Sigma \binom{j+s}{s}\binom{j+s+v}{v} (iD)^s H_{a_j} (iD)^v a_k \\ -\Sigma \binom{j+s}{j}\binom{k}{v} (-1)^v a_k (iD)^{v+s} H_{a_j} \end{bmatrix} \cdot \xi^{-j-1-s-v+k}$$

(with $-j - 1 - s - v + k = r \geq 0$, $v + s = \mu$)

$$= \sum_{r=0}^{m-2} \begin{bmatrix} \Sigma \binom{j+\mu}{\mu}\binom{\mu}{s} (iD)^s H_{a_j} \cdot (iD)^{\mu-s} a_{r+j+\mu+1} \\ -\Sigma \binom{r+j+\mu+1}{v}\binom{j+s}{s} (-1)^v a_{r+j+\mu+1} (iD)^\mu H_{a_j} \end{bmatrix} \xi^r =$$

$$\sum_{r=0}^{m-2} \begin{bmatrix} \Sigma \binom{j+\mu}{\mu} (iD)^\mu \quad (H_{a_j} \cdot a_{r+j+\mu+1}) \\ -\Sigma \binom{r+\mu}{\mu} a_{r+j+\mu+1} (-iD)^\mu H_{a_j} \end{bmatrix} \cdot \xi^r = \sum_{r=0}^{m} \cdot \dot{a}_r \xi^r , \tag{3.21}$$

where the sum in the brackets is understood to extend over all nonnegative integers with the proviso that only terms with a_j, $j = 0,1,\ldots,m$, can appear. This formula agrees with the formula of Gelfand-Dikii [1] if we set $a_m = 1$, $a_{m-1} = 0$, i.e. in the setting of Φ_m, (3.6), and so Theorem 3 is proven.

Remark 3. In fact in [1], Gelfand-Dikii arrived at this formula from computational considerations which shall become apparent in Theorem 4, and then proceeded to verify by a difficult computation that (3.21) in fact yielded a Poisson bracket via

$$\{G,H\} \equiv \left.\frac{dG}{dt}\right|_H , \quad \text{with } H \text{ the Hamiltonian in (3.21)} .$$

Note (3.21) is equivalent to stating that the Hamiltonian vector field associated with the function H is

$$X_H = \sum_{i=0}^{m-2} \left[(J) \cdot \frac{DH}{Da}\right]_i \cdot \frac{D}{Da_i} ,$$

where $\frac{D}{Da} = \left(\frac{D}{Da_0}, \frac{D}{Da_{m-2}}\right)^T$, with

$$J = \Delta - \Delta^* ,$$

where Δ is the $(m-1) \times (m-1)$ matrix differential operator with components

$$\Delta_{rs} = \sum_\gamma \binom{\gamma+r}{r} a_{\gamma+1+r+s}(-iD)^\gamma, \quad 0 \le r \le m-2, \quad 0 \le s \le m-2 ,$$

and Δ^* is the dual of Δ with respect to the natural inner product on $R_n^{(m-1)}$, $(v,w) \doteq \sum_{i=0}^{m-2} \overline{v_i w_i} \in I$, and the remark after (3.21) concerning summation applies above. Clearly for $m = 2$, $X_H = -2iD \cdot \frac{D}{Da_0}$, the well-known Poisson structure of Gardner associated with the Korteweg-deVries equation. Equivalently the Poisson bracket

$$\{G,H\} \doteq X_H G \doteq \left(\Delta\left(\frac{DH}{Da}\right), \frac{DG}{Da}\right) - \left(\frac{DH}{Da}, \Delta\left(\frac{DG}{Da}\right)\right) .$$

Remark 4. It is interesting to study the orbit of $\xi^m + \sum_{i=0}^{m-2} a_i \xi^i$ under the group G, in fact we claim that $\overline{a_{m-2}}$, $\bar{a}_{m-3}$ are always orbit invariants. Since a typical element of G is of the form $1 + x$, $x \in A_{-\infty,-1}$, i.e. $x = \sum_{j\ge1} x_j \xi^{-j}$, to prove the remark we compute, (using (3.1)), that under the action of G

$$\delta A = ((1-x)A(1+x+x^2+\cdots)]_+ - A = [[A,x]_+ \cdot \{1+x+x^2+\cdots\}]_+$$

$$= (C_1 D x_1)\xi^{m-2} + (C_2 x_1 D x_1 + C_3 D^2 x_1 + C_4 D x_2)\xi^{m-3} + \eta,\ \eta \in A_{0,m-4}\ ,$$

with C_1, C_2, C_3, C_4, constants depending on m, and so δa_{m-2}, $\delta a_{m-3} \subset DR$, thus proving the assertion. In fact more is true, namely $\langle A^{\frac{s}{m}}\rangle$, $1 \leq s \leq m-1$ are also orbit invariants, which in fact implies the above assertion. (See Corollary 1 at the end of this section.) Thus in the case of the Korteweg-deVries equation, an orbit of $-D^2 + q$ is specified by $\bar{q}$ = constant. Thus the apparent one-dimensional degeneracy of the Gardner structure is really a consequence of the representation of an orbit by q, which really contains a hidden relation, $\bar{q}$ = constant. This phenomena is well known in the Toda system, where as previously discussed in Section 2, one has the similar orbit relation, trace A = constant.

Remark 5. Suppose we had instead defined

$$\Phi_n = \{\xi^m + \sum_{j=0}^{m-2} (a_j \xi^j + (\xi - iD)^j a_j) \mid a_j \text{ a generator of } R(a)\},\quad m < \infty\ ,$$

i.e. we coordinize Φ_m by these a_j's. Then as our arguments have shown (3.18) defines a Hamiltonian structure. Of course we have to compute H_A, as in (3.19), or equivalently, we must compute the dual of the $\eta_j = \xi^j + (\xi - iD)^j$, $0 \leq j \leq \infty$ in $\mathcal{L}$. The computation of the dual of ξ^j depended on inverting the first relation in (3.14), but in fact, it was only necessary to invert the linear, strictly triangular relation up to degree $m + 1$ to compute the Hamiltonian structure from (3.18). This is always a trivial matter, since m is finite, and the relation to be inverted is upper triangular. Hence we can use the above remarks to compute the Hamiltonian structure of Φ_m in the coordinates of the 'formally self-adjoint operators' i.e. with the above coordinization of Φ_m. In fact, all the results of this paper have and shall be independent of which choice of coordinates one uses on Φ_m, although for clarity we shall sometimes just work with the original coordinates on Φ_m. However, in the case of the formally self-adjoint operators, we have extra reality conditions, if we require the a_j's to be real, for instance the Poisson bracket and traces, (3.10) must be

real. Hence for the rest of the paper, we shall mean either of the two coordinatizations by our expression ϕ_m, unless otherwise stated. For the 'self-adjoint' case, we have for J, the symplectic matrix operator, for $m = 3$, $J = \begin{bmatrix} 0, D \\ D, 0 \end{bmatrix}$, which applies to the Boussinesq equation, see [5], [6]. For $m = 4$, one can show

$$J = \begin{bmatrix} k & 0 & 0 \\ 0 & -1/2D & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad k = -1/2D^3 + a_2 D + Da_2, \quad \text{see [5]} .$$

Remark 6. If we had instead chosen $R \to \tilde{R} = \tilde{R}(\ldots, a_k^{(ij)}, \ldots) = ML(n,R)$, the ring of $n \times n$ matrices over R, instead of R to be the base ring of our construction, then we would have to redefine for $\phi \in \Phi$, (see (3.3)), $\operatorname{tr} \phi \doteq$ matrix trace $\bar{a}_{-1} \in I$, which extends definition (3.3). Note I is still R/DR. All the constructions and arguments of this paper apply to this case. For instance $(E_{ij}\xi^k \cdot)^*$, where E_{ij} is the matrix with its i,j component unity, all other elements zero, equals $(\xi - iD)^{-k-1} \cdot E_{ji}$, and 1 shall often denote the $n \times n$-identity matrix, hence H_{a_k} is the matrix such that $[H_{a_k}]_{j\ell} = \dfrac{DH}{Da_k^{(\ell j)}}$ which would have to be substituted in (3.21). In addition, one may specialize the coefficients of A to lie in some ring of matrices, which amounts to changing the base ring to that ring of matrices.

We now pursue the result analogous to Theorem 1, but first some brief observations. Suppose $H \doteq \langle A^{\nu} \rangle$, $\nu = \frac{N}{n}$, N an integer. We wish to compute H_A. We first write $A = [\lambda^{\nu}\phi_{\lambda}]_{(\lambda,\xi^n)}$, by (3.9), and so $H \doteq \langle [\lambda^{\nu}\phi_{\lambda}]|_{(\lambda,\xi^n)} \rangle$. In addition, by (3.7) we have $\delta\varphi_{\lambda} = \phi_{\lambda} \circ \delta A \circ \phi_{\lambda}$, $\dfrac{d\varphi_{\lambda}}{d\lambda} = -\varphi_{\lambda}^2$, with δ indicating an increment, and so

$$\delta H \doteq \langle [\lambda^{\nu}\phi_{\lambda} \circ \delta A \circ \phi_{\lambda}]_{(\lambda,\xi^n)} \rangle \doteq \langle [\lambda^{\nu}\phi_{\lambda}^2]|_{(\lambda,\xi^n)}, \delta A \rangle + \langle \partial_{\xi}\{\cdot\} \rangle$$

$$= \langle [\lambda^{\nu}\phi_{\lambda}^2]_{(\lambda,\xi^n)}, A \rangle = \left\langle \left[-\left(\frac{\partial\phi_{\lambda}}{\partial\lambda}\right)\lambda^{\nu} \right]_{(\lambda,\xi^n)}, \delta A \right\rangle$$

$$= \left\langle \left[\frac{\partial\lambda^{\nu}}{\partial\lambda} \cdot \phi_{\lambda} \right]_{(\lambda,\xi^n)} + [\partial_{\lambda}\{\cdot\}]_{(\lambda,\xi^n)}, \delta A \right\rangle = \langle [\nu\lambda^{\nu-1}\phi_{\lambda}]_{(\lambda,\xi^n)}, \delta A \rangle = \langle \nu A^{\nu-1}, \delta A \rangle ,$$

(when we have repeatedly used that a perfect derivative can have no residue term), hence we have

$$\langle A^{\nu}\rangle_A = (\nu A^{\nu-1})_{-(m-1)}, \qquad A \in \Phi_m, \tag{3.22}$$

where the operation $(\ )_{-n}$ denotes the natural projection into $\tilde{A}_{-n,-1}$ by $\tilde{P}_{-n,0}$.

We next compute with $\nu = \frac{N}{n}$, $H \doteq \langle A^{\nu}\rangle \doteq \langle [\lambda^{\nu}\phi_{\lambda}]_{(\lambda,\xi^m)}\rangle \doteq$ (interchanging residue evaluations, see (3.3)) $-[\lambda^{\nu}[\phi_{\lambda}]_{(\xi,\lambda^{1/m})}]_{(\lambda,\infty)} \doteq -[\lambda^{\nu}K(\phi,\lambda)]_{(\lambda,\infty)} \doteq \sigma\cdot\mu_{N+1}$, by (3.11). In this calculation we used the meaningful convention that the residue term at ∞ is $-\lambda^{-1}$. One can justify the interchange of residue computations in the above by thinking of performing the above in some analytic setting under appropriate restrictions, where the residue calculations are just achieved by contour integrations in λ,ξ in a small sector near ∞, and the results in this setting imply formal identities which yields the above formal result. We have thus shown

<u>Lemma 1</u>.

$$K(\varphi,\lambda) \doteq \sigma\cdot\sum_{\ell>0}\langle A^{\frac{\ell-1}{m}}\rangle(\lambda^{\frac{1}{m}})^{-(m+\ell-1)} \doteq \sigma\cdot\langle\{\lambda[(\tfrac{A}{\lambda})^{\frac{1}{m}} - (\tfrac{A}{\lambda})^{\frac{2}{m}}]\}^{-1}\rangle, \tag{3.23}$$

$$\langle A^{\frac{N+m}{m}}\rangle_A = \left(\frac{N+m}{m}\right)(A^{\frac{N}{m}})_{-(m-1)}. \tag{3.24}$$

Note that certainly the computational equivalents of Lemma 1 can be found in Gelfand-Dikii, [1] but of course proved and viewed in a different fashion. Note that (3.23) implies that the λ^{-j}, $j = 1,2,\ldots$ terms of $K(\phi,\lambda)$ are missing, which is no surprise. We can now prove the analog of Theorem 1, namely:

<u>Theorem 4</u>. (Gelfand-Dikii). If $H = H^N = \frac{m}{N+m}\langle A^{\frac{N+m}{m}}\rangle$, $A \in \Phi_m$ then Hamiltons equations, $\dot{A} = [H_A, A]^{m-2}$, imply the Lax-isospectral equation

$$\dot{A} = [A, P_N], \text{ with } P = P_N = (A^{\frac{N}{n}})_+, \tag{3.25}$$

with $(\ )_+$ as usual indicating projection onto $A_{0,\infty}$. The H_N is $\frac{m}{N+n}$ times the coefficient of the $(\lambda^{\frac{1}{m}})^{-(N+2m)}$ term of the 'trace' of the resolvent, $K(\phi,\lambda)$. In

addition the $\langle A^{\frac{N}{m}}\rangle$ are in involution with respect to $\{\ ,\ \}$, the Poisson bracket.

Proof: Theorem 4, like Theorem 1, depends on the crucial observation $[A^s,A] = 0$ for all s. This is a consequence of the definition of $f(L)$ by residues (3.9) and the well-known functional equation of the resolvent $\frac{\phi_{\lambda_1} - \phi_{\lambda_2}}{\lambda_1 - \lambda_2} = \phi_{\lambda_1} \circ \phi_{\lambda_2}$, which implies $A^s \cdot A^t = A^{s+t}$. Clearly $A^{\frac{N}{m}} = (A^{\frac{N}{m}})_+ + (A^{\frac{N}{m}})_-$, with $(\ )_-$ denoting projection into $\mathcal{A}_{-\infty,-1}$, and so $0 = [A^{\frac{N}{m}},A] = [(A^{\frac{N}{m}})_+ + (A^{\frac{N}{m}})_-,A]$, therefore $[(A^{\frac{N}{m}})_-,A] = [A,(A^{\frac{N}{m}})_+]$ and so we have

$$[A,(A^{\frac{N}{m}})_+] = [(A^{\frac{N}{m}})_{-(m-1)},A] + [\theta_m,A] \ , \tag{3.26}$$

where θ_{-m} is contained in $\tilde{\mathcal{A}}_{-\infty,-m}$. We project (3.26) onto $\mathcal{A}_{0,\infty}$, observing that $[A,(A^{\frac{N}{m}})_+] \in \mathcal{A}_{0,\infty}$, while $[\theta_{-m},A] \in \mathcal{A}_{-\infty,-1}$, $[(A^{\frac{N}{m}})_{-(m-1)},A] \in \mathcal{A}_{-\infty,m-2}$. We thus after projection onto $\mathcal{A}_{0,\infty}$, arrive at

$$[A,(A^{\frac{N}{m}})_+] = [(A^{\frac{N}{m}})_{-(m-1)},A]^{m-2}, \text{ which is a member of } \mathcal{A}_{0,m-2} \ ,$$

and thus as a consequence of Lemma 1, Theorem 4 is proven, modulo the involution statement.

We now prove the involution statement, namely that

$$\{\langle A^{\frac{N}{m}}\rangle,\langle A^{\frac{K}{m}}\rangle\} = 0, \ N,K \text{ integers} \ ,$$

where $\{\ ,\ \}$ is the Poisson bracket, (3.20), i.e. we show that the traces of A are in involution. If $H = \langle A^{\frac{K}{m}}\rangle$, then $\{\langle A^{\frac{N}{m}}\rangle,\langle A^{\frac{K}{m}}\rangle\} = \frac{d}{dt}\langle A^{\frac{N}{m}}\rangle$, by the Hamiltonian formalism, with $H = \langle A^{\frac{K}{n}}\rangle$, i.e.

$$\{\langle A^{\frac{N}{m}}\rangle,\langle A^{\frac{K}{m}}\rangle\} = \langle \dot{A},\langle A^{\frac{N}{m}}\rangle_A\rangle, \text{ by the definition of gradient} \ ,$$

$$= \langle [A,P_K],(A^{\frac{N}{m}})_{-(m-1)}\rangle = \langle [A,P_K],(A^{\frac{N}{m}})_-\rangle, \quad (\text{since } [A,P_K] \in \mathcal{A}_{0,m-2}, \text{ see } (3.16))$$

$= \langle [A,P_K]\,(A^{\frac{N}{m}})_- \rangle = \langle [A,P_K]A^{\frac{N}{m}} \rangle$, (since the trace only picks out the ξ^{-1} term)

$= \langle P_K, [A^{\frac{N}{m}}, A] \rangle = 0$, thus completing the proof of Theorem 4.

We wish to point out the parallels between the proofs of Theorems 1, 4. We also wish to remind the reader of Remarks 5, 6 which imply that Theorem 4 really constitutes a generalization of the theorem of Gel'fand-Dikii, as it includes the cases discussed in these remarks, with but minor modifications in terminology.

<u>Corollary 1</u>. If $A \in \Phi_m$, $\langle A^{\frac{s}{m}} \rangle$, $s = 1,2,\dots,m-1$, are orbit invariants, which in fact implies $\bar{a}_{m-2}$, $\bar{a}_{m-3}$ are orbit invariants.

<u>Proof</u>: Since $P_N = (A^{\frac{N}{m}})_+$, $P_N = 0$ for N a negative integer, hence $[A,P_N] = 0$, and so in Theorem 4, $\dot{A} = 0$, hence $\dot{A} = [H_A^{N'}, A]^{m-2} = 0$ for $N' < m$. But $[H_A, A]^{m-2} = 0$ is a necessary and sufficient condition for a function H to be an orbit invariant, by (3.20), and thus we have shown that $\langle A^{\frac{s}{n}} \rangle$, $s = 1,2,\dots,m-1$, are orbit invariants.

To see that $\bar{a}_{m-2}$, $\bar{a}_{m-3}$ are orbit invariants, observe that $A = \xi^m + \sum_{i=0}^{m-2} a_i \xi^i$ has weight m if we assign ξ weight 1, and a_i weight $m - i$. From the definition $A^{\frac{N}{m}}$, (3.9), it is easy to see $A^{\frac{N}{m}}$ has weigh N if we further define $D^j a_i$ to have weight $j + (m-i)$, i.e. D to have weight one. Thus if $\langle A^{\frac{N}{m}} \rangle = \bar{E}_N$, E_N must have weight $N + 1$, and so we must have

$$\langle A^{\frac{1}{m}} \rangle = C_1 \bar{a}_{m-2}, \qquad \langle A^{\frac{2}{m}} \rangle = C_2 \bar{a}_{m-3},$$

C_1, C_2 being (nonzero) constants which can be computed explicitly. We conjecture that the algebra formed by the invariants of the corollary provide a complete list of orbit invariants, at least those which can be expressed as I elements.

4. Lenard Relations

We are, from the considerations of the previous two sections, in a position to compute recursion relations, or so-called Lenard relations for various quantities of interest, see [5] for instance, where such relations are discussed.

We consider the setup in Section 3. The crucial tool in these relations is the identity $A^s \cdot A^t = A^{s+t}$, which plays such an important role in Theorems 1, 4 in the weaker form $[A^s, A^1] = 0$, and whose proof was discussed in the proof of Theorem 4. In addition the general relation $\langle [E,P] \rangle = 0$ is important (see Theorem 2).

We make the usual natural decomposition of Φ associated with $\mathcal{L}, \mathcal{L}^*$, namely $B \in \Phi$ implies $B = B_+ + B_-$, with $B_+ \in \mathcal{L}^* = A_{0,\infty}$, $B_- \in \mathcal{L} = A_{-\infty,-1}$. We thus have, with $A \in \Phi_n$,

$$(A^{\frac{N}{n}+1})_+ = (A^{\frac{N}{n}} \cdot A)_+ = (A^{\frac{N}{n}})_+ \cdot A + ((A^{\frac{N}{n}})_- \cdot A)_+ ,$$

however,

$$(A^{\frac{N}{n}})_- = (A^{\frac{N}{n}})_{-(n-1)} + \theta_{-n} = H_A^N + \theta_{-n} ,$$

with $\theta_{-n} \in \tilde{A}_{-\infty,-n}$, by Lemma 1, using the notation of Theorem 4. The above yields, still using the notation of Theorem 4, (note $P_N = 0$, N a negative integer)

$$P_{N+n} = P_N A + (H_A^N \cdot A)_+ + E_0 , \tag{4.1}$$

with E_0 equaling the coefficient of the $(\xi - iD)^{-n}$ term of $(A^{\frac{N}{n}})_-$, i.e. the coefficient of the component of $[(A^{\frac{N}{n}})_- - H_A^N]$ in $\tilde{A}_{-n,-n}$. In addition, since $\langle [(A^{\frac{N}{n}})_-, A] \rangle \doteq 0$ *by Theorem 2, the* ξ^{-1} *coefficient of* $[(A^{\frac{N}{n}})_-, A]$ *is an exact derivative*, identically in the coefficients of $(A^{\frac{N}{n}})_-, A$. But, by the rule of multiplication, (3.1), the coefficient of the ξ^{-1} term of the bracketed expression is of the form (up to a constant multiple),

$$DE_0 + Q((A^{\frac{N}{n}})_{-(n-1)}) = DE_0 + Q(H_A^N), \text{ by Lemma 1 ,}$$

but since $[A^{\frac{N}{n}}, A] = 0$, hence $[(A^{\frac{N}{n}})_-, A]_- = 0$, both sides of the equation must equal zero. Here $Q(H_A^N)$ just depends on the coefficients of H_A^N and their derivatives. We thus have the term $Q(H_A^N)$ is an exact derivative in the coefficients of H_A^N, i.e. $Q(H_A^N) = DQ'(H_A^N)$, and in fact $Q'(\cdot)$ is easily computable. We thus have $E_0 = -Q'(H_A^N)$, from which we deduce, using (4.1),

$$P_{N+n} = P_N A + \psi(H_A^N), \quad \psi(H_A) = (H_A A)_+ + \frac{iD^{-1}}{n}(P_{0,0}([H_A, A]\xi)) , \tag{4.2}$$

and we note the coefficients of $\psi \in A_{0,\infty}$ are polynomial functions of $\frac{DH}{Da_i}$, $0 \le i \le n-2$, and their derivatives. We view ψ as an operator, $\psi : \tilde{A}_{-(n-1),-1} \to A_{0,n-1}$.

We now bracket (4.2) with A, yielding

$$[A, P_{N+n}] = [A, P_N] \cdot A + [A, \psi(H_A^N)] . \tag{4.3}$$

By Theorem 4, $[A, P_N] = \dot{A} \equiv H(J\frac{DH^N}{Da})$, with $J = \Delta - \Delta^*$ the matrix differential operator which determines Hamilton's equation (see Section 3), and H defined in the obvious way, i.e. $H\begin{pmatrix} a_0 \\ \vdots \\ a_{n-2}\end{pmatrix} = A - \xi^n$. Also note that the right hand side of (4.3) depends only on H_A^N, by Theorem 4, and is contained in $A_{0,n-2}$ since the left hand side is. We can thus define the $(n-1) \times (n-1)$ matrix differential operator M by

$$H(M\,\frac{DH^N}{Da}) \equiv H(J\,\frac{DH^N}{Da}) \cdot A + [A, \psi(H_A^N)] , \tag{4.4}$$

and since $[A, P_{N+n}] = J(\frac{DH^{N+n}}{Da})$, (4.3), (4.4) implies

$$M(\frac{DH^N}{Da}) = J(\frac{DH^{N+n}}{Da}) , \tag{4.5}$$

the standard form of Lenard recursion relations, since M, J are matrix differential operators. Equation (4.2) is equivalent to

$$P_{s+nt} = \sum_{\nu=0}^{t} \psi_{\nu,s} A^{t-\nu}, \quad \psi_{\nu,s} = \psi(H_A^{(\nu-1)\cdot n+s}), \quad 0 \le s \le n-1 . \tag{4.6}$$

We observe that (4.4), (4.5), (4.6) are enough to establish $[A, P_N] = H(JH_A^N)$, providing we define $P_N = 0$, $N < 0$, and we shall think of these as the Lenard relations.

One computes $M(\frac{DH}{DA}) = A(H_A A)_+ - (AH_A)_+ A + \frac{1}{n}\sum_{v=1}^{v=n} \frac{1}{v!}(\frac{\partial}{\partial\xi})^v A \cdot (-iD)^{v-1}(P_{0,0}([H_A, A]\xi))$.

Also note equation (4.5) immediately implies Corollary 1, Section 3.

Remark 7. The self-adjoint case, and for that matter the case of Remark 6, may be done in precisely the same way as above, but has some novel features. For if $A = \xi^n + \sum_{j=0}^{n-2} (a_j\xi^j + (\xi - iD^j a_j)$, a_j real, then if ψ is a formal eigenvector with eigenvalue λ of the associated differential operator, we have by the self-adjointness of A,

$$\langle \psi\bar{\psi}\rangle \cdot \frac{D\lambda}{Da_j} = [((iD)^j\psi)\psi^* + \psi((iD)^j\psi)^*], \qquad j = 0,1,\ldots,n-2, \tag{4.7}$$

where $*$ denotes complex conjugation. And so (4.5) implies

$$M\left(\widetilde{\frac{D\lambda}{Da}}\right) = \lambda J\left(\widetilde{\frac{D\lambda}{Da}}\right), \qquad \left(\widetilde{\frac{D\lambda}{Da}}\right) = \langle\psi\bar{\psi}\rangle \frac{D\lambda}{Da}. \tag{4.8}$$

Note that since $\langle\psi\psi^*\rangle \in I$, it plays the rate of a constant in (4.8), (see beginning of Section 3).

Remark 8. One conjectures from computational evidence that M in fact defines a symplectic structure, like its counterpart J. One also conjectures that (4.4), is in fact a functional identity in its arguments, i.e. (4.4) doesn't depend for its validity upon the substitution of H_A^N into its argument. In addition, equations (4.4), (4.5) have a bias, at least in their derivations, for instance if instead of using $A^{\frac{N}{n}+1} = A^{\frac{N}{n}} \cdot A$, we had employed $A^{\frac{N}{n}+1} = A \cdot A^{\frac{N}{n}}$, we would have certainly gotten a different ψ, but we doubt that we would have gotten a different M; however, we cannot prove this, and merely conjecture this to be so.

Examples. For the self-adjoint case, $A = -D^2 + a_0$, we have

$$J = D, \quad M = -\frac{1}{4}\{D^3 - 2(a_0 D + Da_0\ \cdot)\}, \quad \psi(v) = \frac{1}{4}(2vD - (Dv)).$$

These formulas are certainly well known. For

$$A = iD^3 + i(a_1 D + Da_1\cdot\ ) + a_0,$$

we have

$$\mathcal{J} = \begin{bmatrix} 0, & D \\ D, & 0 \end{bmatrix},$$

$$\mathcal{M} = \begin{bmatrix} M_1, & M_2 \\ M_3, & M_4 \end{bmatrix}$$

$$M_1 = \frac{1}{3} [D^3 + a_1 D + Da_1 \cdot)$$

$$M_2 = a_0 D + \frac{2}{3} (Da_0), \qquad M_3 = a_0 D + \frac{1}{3} (Da_0)$$

$$M_4 = \frac{1}{9} [D^5 + 5a_1 D^3 + D^3 (5a_1) \cdot + (8a_1^2 - 3(D^2 a_1))D + D(8a_1^2 - 3(D^2 a_1)) \cdot] ,$$

$$\psi \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \frac{1}{3} \{2iv_2 D^2 + (v_1 - i(Dv_2))D + \frac{i}{3} (8a_1 v_2 + (D^2 v_2 - Dv_1)\} .$$

These formulas for $\mathcal{J}$, $\mathcal{M}$ have been found independently by H. McKean [6]. For

$$A = D^4 + (a_2 D^2 + D^2 a_2 \cdot) + i(a_1 D + Da_1 \cdot) + a_0 ,$$

then

$$J = \begin{bmatrix} J_{11}, & J_{12}, & J_{13} \\ J_{21}, & J_{22}, & J_{23} \\ J_{31}, & J_{32}, & J_{33} \end{bmatrix}, \qquad \mathcal{M} = \begin{bmatrix} M_{11}, & M_{12}, & M_{13} \\ M_{21}, & M_{22}, & M_{23} \\ M_{31}, & M_{32}, & M_{33} \end{bmatrix},$$

where

$$J_{11} = J_{12} = J_{21} = J_{23} = J_{32} = 0 ,$$

$$J_{13} = J_{31} = D, \quad J_{22} = -\frac{1}{2} D, \quad J_{33} = -\frac{1}{2} D^3 + (a_2 D + Da_2 \cdot),$$

and letting $D^2\psi = \psi''$, etc., we have

$$M_{11} = \frac{5}{8} D^3 + \frac{1}{4} (a_2 D + Da_2 \cdot), \quad M_{12} = -M_{21}^{*} = \frac{3}{4} a_1 D + \frac{1}{2} a_1' ,$$

$$M_{13} = -A_{31}^{*} = -\frac{1}{4} D^5 + \frac{1}{4} a_2 D^3 + \frac{5}{4} a_2' D^2 + (\frac{7}{4} a_2'' + a_0)D + (\frac{3}{4} a_2''' + \frac{3}{4} a_0') ,$$

$$M_{22} = \frac{1}{2} \{\frac{1}{4} D^5 + \frac{1}{2} [a_2 D^3 + D^3 a_2] - \frac{1}{2} [(a_2'' + a_0 - a_2^2)D + D(a_2'' + a_0 - a_2^2) \cdot]\} ,$$

$$M_{23} = -A^{\star}_{32} = -a_1 D^3 - \frac{3}{2} a_1' D^2 - (a_1'' + \frac{1}{2} a_2 a_1) D - (\frac{1}{2} a_1' a_2 + \frac{1}{4} a_1''') ,$$

$$M_{33} = \frac{1}{8} D^7 + \frac{1}{2} [(a_2^2 - a_2'' - \frac{1}{2} a_0) D^3 + D^3 (a_2^2 - a_2'' - \frac{1}{2} a_0) \cdot]$$

$$+ [(\frac{1}{4} a_2'''' - a_2'^2 + \frac{3}{4} a_1^2 + a_2 a_0) D + D(\frac{1}{4} a_2''' - a_2'^2 + \frac{3}{4} a_1^2 + a_2 a_0)]$$

while

$$\psi(v) = \phi \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \frac{1}{2} v_3 D^3 - \frac{1}{4} (v_3' + i v_2) D^2 + [\frac{1}{4} (v_1 + i v_2') + \phi_1 v_3] D$$

$$+ \frac{1}{4} [\frac{1}{2} (-3v_1' - i v_2'' + v_3''') - 2\phi_1 v_3' + 3i\phi_2 v_3 - i\phi_1 v_2] .$$

Clearly things rapidly get out of hand.

We note that these formulas are computed from (4.1)-(4.4). In the case $A = \xi^n + \sum_{i=0}^{n-2} a_i \xi^i$, one can use the above formula to get general formulas for M, J, ψ, but since in the self-adjoint case, one must compute the dual of $(\xi - iD)^j + \xi^j = \eta_j$ by hand (up to order ξ^{-n-1}), one computes M, J, ψ separately for each n. In [8], Symes computes M, J for the case $A = \xi^3 + a_1 \xi + a_0$.

Remark 9. The relations of this Section also apply to the systems of Section 2. We derive relation (4.2) for these systems which imply relations of the form (4.4), (4.5), (4.6).

We shall use the notation of Theorem 1. Observe

$$L^N = (L^N)^+ - (L^N)^- + (L^N)^0 + 2(L^N)^- = P_N + B_N ,$$

where

$$P_N = (L^N)^+ - (L^N)^- ,$$

$$B_N = (L^N)^0 + 2(L^N)^- = (H_N)_A + \theta_m ,$$

by (2.14), where $H = \langle \frac{1}{N+1} A^{N+1} \rangle$, and $\theta_m \in A_{-n,-m-1}$. From this we conclude

$$P_{N+1} = L^{N+1} - B_{N+1} = (P_N + B_N) \cdot L - B_{N+1}$$

$$= P_N L + (B_N \cdot L) - B_{N+1}$$

$$= [P_N L + (H_N)_A \cdot L] + [\theta_m \cdot L - B_{N+1}] ,$$

hence, remembering $(P_{N+1})^+ \in A_{1,n}$, we have shown

$$(P_{N+1})^+ = (P_N L)^+ + [(H_N)_A \cdot L]^+ \equiv (P_N L)^+ - \Delta((H_N)_A) . \tag{4.9}$$

The above defines the function $\Delta(\cdot)$. Taking the negative transpose of the above equation, we have $P^-_{N+1} = (LP_N)^- + \Delta^T((H_N)_A)$, where we have used $P^T_N = -P_N$, $L^T = L$, but

$$(P_N L)_{-n} - (LP_N)_{-n} = [P_N, L]_{-n} = ([A, H^N_A]_+)^T, \quad \text{(by Theorem 1)} , \tag{4.10}$$

and so (4.9), (4.10), and the expression for P^-_{N+1}, imply

$$P_{N+1} = P_N \cdot L + \tilde{\psi}((H_N)_A) ,$$

with $\tilde{\psi}(H_A) = (H_A L)^+ - (LH^T_A)^- + ([H_A, A]^+)^T$. Thus a relation of the form (4.2) has been established, as claimed. We also note that the conjectures of Remark 8 are relevant in this case.

We give the results for the Toda system, i.e. $A_{i,i} = b_i$, $A_{i,i+1} = a_i$, all other $A_{i,j}$'s equal to zero. Define $E_{ij} = 1$ if $j = i + 1$, zero otherwise and similarly for E^T_{ij}. Let M, J act on in general the vector $\begin{pmatrix} H_a \\ H_b \end{pmatrix}$, $H = H(a,b)$, $H_a = \left(\frac{\partial H}{\partial a_0}, \dots, \frac{\partial H}{\partial a_{n-1}} \right)^T$, etc. for H_b. Then we have

$$J = \begin{bmatrix} O_n, & S \\ -S^T, & O_{n+1} \end{bmatrix} ,$$

O_n, O_{n+1} the $n \times n$, and $(n + 1) \times (n + 1)$ zero matrices respectively, and $[S]_{ij} = (-\delta_{ij} + E_{ij})a_i$, $i = 0,\dots,n - 1$, $j = 0,\dots,n$.

$$M = \begin{bmatrix} M_1, & M_3 \\ -M_3^T, & M_2 \end{bmatrix}, \qquad \text{where}$$

$$[M_1]_{ij} = \frac{a_i a_{i+1}}{2} (E_{ij} - E_{ij}^T), \quad i,j = 0,\ldots,n-1,$$

$$[M_2]_{ij} = 2a_i^2 (E_{ij} - E_{ij}^T), \quad i,j = 0,\ldots,n,$$

$$[M_3]_{ij} = a_i(b_{i+1}E_{ij} - b_i\delta_{ij}), \quad i = 0,\ldots,n-1, \quad j = 0,\ldots,n.$$

Finally if $w = (u_0,\ldots,u_{n-1},v_0,\ldots,v_n)^T$, then we have

$$[\tilde{\psi}(\mathbf{w})]_{ij} = \frac{1}{2}\delta_{ij}(a_i u_i - a_{i-1}u_{i-1}) - a_i(E_{ij}v_i - E_{ij}^T v_{i+1}), \quad i,j = 0,1,\ldots,n.$$

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This paper developed out of an attempt to understand results of Gelfand-Dikii ~~[1]~~ and P. Moerbeke ~~(unpublished version of [2])~~, in a unified way. ~~We provide~~ a natural abstract setting is provided for understanding ~~and~~ the symplectic structure involved in both results. The setting is an orbit in the dual algebra of a group. ~~We also discuss~~ a natural trace functional for formal asymptotic pseudo-differential operators, ~~In addition we discuss~~ as well as so-called Lenard recursion relations inherent in these structures are discussed.

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